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VERTICAL VIBRATION OF A RIGID CIRCULAR BODY AND HARMONIC ROCKING OF A RIGID RECTANGULAR BODY ON AN ELASTIC STRATUM

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Abstract—The mixed boundary-value problems of the vertical vibration of a rigid circular body and the rocking of a long rigid rectangular body on an infinitely wide elastic stratum have been precisely formulated in terms of dual integral equations. Approximate solutions of these equations for the case of a frictionless foundation base have been obtained by establishing in a novel manner an equivalent system on a semi-infinite elastic medium. It is shown that the response of a body vibrating at frequency factor η_2 on a stratum of finite depth is approximately equivalent to that of the body with its inertia increased by a factor η_2^2/η_{2e}^2 but vibrating at a lower frequency factor $\eta_{2e} = (\eta_2^2 - 1/\bar{h}^2)^{\frac{1}{2}}$ on a semi-infinite medium of the same elastic constants as the stratum of non-dimensional depth $\bar{h}$. All the results approach corresponding semi-infinite medium results as the stratum depth tends to infinity. This, therefore, corrects the error of Warburton [1] in which the response of a body on a semi-infinite medium lies between responses on strata of finite depths contrary to the expected asymptotic approach confirmed by the experiments of Arnold *et al.* [2].

Finally, two important results are established for this system: a stratum depth of about five times the base radius (or semi-width, for the rectangular body) is a very fair approximation to a semi-infinite medium; resonant frequency of a body on a stratum decreases with increasing stratum depth. Furthermore, the resonant frequency factor, η_2, of bodies with large inertia ratios (greater than about 10) can be estimated from the semi-infinite medium solution irrespective of the stratum depth. The present theory consistently shows good agreement with the experimental results of Arnold *et al.* [2].

1. INTRODUCTION

THE present work is an attempt to answer two fundamental questions: what depth of a stratum is a fair approximation to a semi-infinite medium when a rigid circular body performs vertical vibrations or a rigid rectangular body rocks on an infinitely wide stratum on a frictionless foundation? Secondly, how does the resonant frequency factor of the body vary with increasing stratum depth?

The practical need to study vibrations of rigid bodies on a stratum rather than the conventional half-space and the review of the slim literature in this field have been more fully discussed by the author in an earlier work [3] which deals with the simplest of the modes of vibration: torsional oscillations of a rigid circular body on an infinitely wide elastic stratum. This work gives, for that simple mode of vibration, the answer to our first question as a stratum depth of about five times the radius. The result is confirmed by Gladwell [4] in a publication later in that year where he considers the same simple case of torsional oscillations and obtains approximate solutions for low frequency factor and high stratum depth ratios using a method devised by Noble for solving Fredholm integral equations of the second kind to which the problem is reduced.

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However, large structures on elastic foundations—like buildings with rotating machinery installed in them—are more frequently excited in translational modes and the need to study both vertical and rocking vibrations, (especially for tall structures) therefore, demands much greater attention.

Unfortunately, Warburton's theory [1] as hinted by the author in the introduction of that earlier work [3] is contrary to practical experience because Figs. 5 and 7 of his paper [1] which considers the problem of the present work shows that the response of a rigid body on an infinitely deep medium lies between responses on strata of finite depths. One can only attribute this error to the fundamental assumption of his theory—using a static stress distribution of the elastic half-space as a starting point for solving a dynamic problem of a stratum.

It is also worth mentioning that Paul in a series of works, for example, [5–7] formulates some analogous problems. The results, however, are far from shedding any light on the answer to the above significant questions.

It is important to emphasize the practical significance of the present investigation. The linear dimensions of the base of large structures, in general, would compare with the depth of the sub-soil for which a reasonably uniform composition can be found. Usually, the modulus of elasticity of the sub-soil increases with depth and it is a far more practical model, for such large structures, to consider a stratum of some depth as affecting the response of the structure to external excitation than the whole depth of the otherwise semi-infinite medium. Thus, in order to have a more accurate result on the resonant frequency of the structure or, conversely, to use the resonant frequency to determine the elastic constants of the medium as in other previous works [8, 9], it is necessary to develop a theory for vibrations on a stratum rather than on an elastic half-space.

We emphasize here a legitimate approach to understanding these problems is to seek first an exact formulation in terms of dual integral equations. Approximate solutions to these equations will give better results than guessing from the start the form of the unknown stress distribution under the rigid body because this is the major factor governing the required response of the body.

The governing dual integral equations are here developed for the first time starting from the governing differential equations of a vibrating elastic medium. We observe later that an interesting and useful reduction of the stratum system to an equivalent semi-infinite medium has been possible by considering an approximation to the governing dual integral equations. This has the main advantage that the behaviour of the stratum problems can be predicted from the elastic half-space models which are now pretty well understood. The present work gives complete results for the semi-infinite medium case of the circular body whilst those for the rocking of a rectangular body have been given in an earlier work [10]. The results lead to answers to the above two questions for the first time on a justifiable theoretical basis and they agree within stated limits of approximation with the experimental results of Arnold *et al.* [2] which, however, are limited only to the case of zero Poisson's ratio.

2. GOVERNING DIFFERENTIAL EQUATIONS AND THEIR GENERAL SOLUTIONS

In the analyses which follow, repetition of previous work is minimized by quoting equations already established by the author in previous works [3, 11]. Also, because of the similarity in the results, only equations for the rigid circular body will be derived whilst corresponding terminal results for the rectangular body will be quoted by attaching letter a to the corresponding equation's number.

It has been shown in [11] that the governing equation for an isotropic elastic medium in the absence of thermal damping,

$$c_1^2 \operatorname{grad} \operatorname{div} \mathbf{s} - c_2^2 \operatorname{curl} \operatorname{curl} \mathbf{s} = \frac{\partial^2 \mathbf{s}}{\partial t^2} \tag{1}$$

reduces to the following

$$\frac{\mathrm{d}^2 \bar{\varepsilon}}{\mathrm{d}z^2} - \alpha_1^2 \bar{\varepsilon} = 0 \tag{2}$$

$$\frac{\mathrm{d}^2 \bar{\omega}_\theta}{\mathrm{d}z^2} - \alpha_2^2 \bar{\omega}_\theta = 0 \tag{3}$$

where

$$\bar{\varepsilon}(p, z) = \int_0^\infty \varepsilon(r, z) r J_0(pr) \, \mathrm{d}r$$

$$\bar{\omega}_\theta(p, z) = \int_0^\infty \omega_\theta(r, z) r J_1(pr) \, \mathrm{d}r$$

are the zero order and first order Hankel transforms of the dilatation $\varepsilon(r, z)$ and component of rotation $\omega_\theta(r, z)$, respectively. The double-valued functions α_1, α_2 are defined as

$$\alpha_1 = \sqrt{(p^2 - \Omega_1^2)}$$

and

$$\alpha_2 = \sqrt{(p^2 - \Omega_2^2)}$$

where $\Omega_1 = \Omega/c_1$, $\Omega_2 = \Omega/c_2$ in which Ω is the angular frequency of sinusoidal vibrations, c_1 and c_2 are the velocities of dilatational and shear waves, respectively.

The general solutions of equations (2) and (3) now take us from the realm of the earlier work. They are given by

$$\bar{\varepsilon} = A_1 \operatorname{sh}(\alpha_1 z) + A_2 \operatorname{ch}(\alpha_1 z) \tag{4}$$

$$\bar{\omega}_\theta = B_1 \operatorname{sh}(\alpha_2 z) + B_2 \operatorname{ch}(\alpha_2 z). \tag{5}$$

We notice the obvious difference between these solutions and the corresponding solutions of the semi-infinite medium where we can set the positive exponential arbitrary functions to zero on the argument that the solution remains finite as the depth coordinate tends to infinity. In the present case, the finiteness of the stratum depth demands the introduction of the two arbitrary functions in each solution. It should also be understood that all stresses and displacements vary sinusoidally with respect to time at an angular frequency Ω.

For the rigid rectangular body, Cartesian coordinates x along the width of the base, y directed to the interior of the medium and z along the infinitely long length coinciding with the axis of rock will be used. The application of complex Fourier transforms leads to precisely the same equations as in (4) and (5) if z is replaced by y and $\bar{\omega}_\theta$ by $\bar{\omega}_z$.

3. EVALUATION OF THE ARBITRARY FUNCTIONS IN TERMS OF BOUNDARY STRESS AND DISPLACEMENT TRANSFORMS

It has been shown in [11] that the relevant transforms of stresses and displacements in the elastic medium are given by

$$\bar{\sigma}_z = \frac{G}{\Omega_2^2}\left\{4p\frac{\mathrm{d}\bar{\omega}_\theta}{\mathrm{d}z}+\beta^2\left[\frac{\lambda}{G}p^2-\beta^2\alpha_1^2\right]\bar{\varepsilon}\right\} \tag{6}$$

$$\bar{\tau}_{rz} = \frac{2G}{\Omega_2^2}\left\{\beta^2 p\frac{\mathrm{d}\bar{\varepsilon}}{\mathrm{d}z}-[p^2+\alpha_2^2]\bar{\omega}_\theta\right\} \tag{7}$$

$$\bar{w} = \frac{1}{\Omega_2^2}\left\{2p\bar{\omega}_\theta-\beta^2\frac{\mathrm{d}\bar{\varepsilon}}{\mathrm{d}z}\right\} \tag{8}$$

where G is the shear modulus and λ the Lame's constant of the medium; $\beta = c_1/c_2$ is the ratio of the wave velocities; w is the component of displacement in the depth direction.

The mixed boundary conditions at the surface are given by

$$\left.\begin{aligned} w(r,0) &= w_0 \quad (0 < r < R) \\ \sigma_z(r,0) &= 0 \quad (r > R) \end{aligned}\right\} \tag{9}$$

We now assume zero shear stress on the whole surface and at the stratum base—the latter corresponding to a frictionless contact with the foundation. In addition the vertical component of displacement vanishes at the base so that the remaining boundary conditions can be expressed as

$$\tau_{rz}(r,0) = 0 \quad (r > 0) \tag{10}$$

$$w(r,h) = 0 \quad (r > 0) \tag{11}$$

$$\tau_{rz}(r,h) = 0 \quad (r > 0) \tag{12}$$

where h is the constant depth of the stratum and R is the radius of the circular base.

It is here observed that a great simplication is achieved by noting from equations (7) and (8) that the boundary conditions in (11) and (12) are readily satisfied by setting each of $\mathrm{d}\bar{\varepsilon}/\mathrm{d}z$ and $\bar{\omega}_\theta$ to zero. Also we replace the mixed boundary conditions (9) by the introduction of a discontinuous stress function valid throughout the surface and incorporating both the unknown stress distribution under the body and the known zero direct stress outside the rigid circular base. The transforms of our boundary conditions, from the foregoing, then take the following alternative form:

$$\bar{\sigma}_z(p,0) = \overline{\sigma(r)} \quad (r > 0) \tag{13}$$

$$\bar{\tau}_{rz}(p,0) = 0 \quad (r > 0) \tag{14}$$

$$\bar{\omega}_\theta(p,h) = 0 \quad (r > 0) \tag{15}$$

$$\mathrm{d}\bar{\varepsilon}/\mathrm{d}z(p,h) = 0 \quad (r > 0) \tag{16}$$

where $\overline{\sigma(r)}$ is the transform of the discontinuous function of the direct stress on the surface of the stratum.

If we now use the solutions in equations (4) and (5) in equations (6)–(8) and set the depth coordinate to zero or h appropriately, we ultimately find that the arbitrary functions are to be determined from the set of simultaneous equations:

$$\beta^2\left(\frac{\lambda}{G}p^2-\beta^2\alpha_1^2\right)A_2+4p\alpha_2B_1 = \frac{\Omega_2^2}{G}\overline{\sigma(r)} \tag{17}$$

$$\beta^2p\alpha_1A_1-(p^2+\alpha_2^2)B_2 = 0 \tag{18}$$

$$\mathrm{sh}(\alpha_2h)B_1+\mathrm{ch}(\alpha_2h)B_2 = 0 \tag{19}$$

$$\mathrm{ch}(\alpha_1h)A_1+\mathrm{sh}(\alpha_1h)A_2 = 0. \tag{20}$$

We shall show in the next section that we require only B_2 to formulate our governing dual integral equations. The evaluation of B_2 from the above equations requires the following preliminary results

$$\frac{\lambda}{G} = \beta^2-2 \tag{21}$$

$$p^2+\alpha_2^2 = 2p^2-\Omega_2^2 \tag{22}$$

$$\frac{\Omega_2^2}{\Omega_1^2} = \beta^2. \tag{23}$$

We then find after some algebra in the determinantal solution of the simultaneous equations that

$$B_2 = \frac{\Omega_2^2p\alpha_1\overline{\sigma(r)}}{G\phi(p,h)} \tag{24}$$

where

$$\phi(p,h) = (2p^2-\Omega_2^2)^2\coth(\alpha_1h)-4p^2\alpha_1\alpha_2\coth(\alpha_2h) \tag{25}$$

is a modified Rayleigh function appropriate to the stratum problem and clearly reduces to the well known Rayleigh function of the semi-infinite medium as h tends to infinity.

It is sufficient to state that the boundary conditions and procedure for evaluation are similar for the rectangular body and, corresponding to equation (24), we have

$$B_2 = -\frac{i\Omega_2^2p\alpha_1\overline{\sigma(x)}}{G\phi(p,h)} \tag{24a}$$

where $\overline{\sigma(x)}$ is the complex Fourier transform of the unknown dynamic stress distribution under the rocking base of semi-width b and $\phi(p, h)$ is exactly the same as in equation (25).

4. FORMULATION OF THE EXACT GOVERNING DUAL INTEGRAL EQUATIONS

We now focus attention on the exact boundary conditions at the surface of the stratum as expressed by equations (9). We find from the first of these equations that we require the transforms of the components of displacement at the surface. Using the solutions of $\bar{e}$ and $\bar{\omega}_\theta$ of equations (4) and (5) in the general solution of $\bar{w}$ in equation (8) and then setting z to zero, we find that

$$(\bar{w})_{z=0} = \frac{1}{\Omega_2^2}(2pB_2-\beta^2A_1\alpha_1). \tag{26i}$$

We eliminate A_1 from this equation using equation (18) so that, after a little algebra, we obtain the simple result

$$(\bar{w})_{z=0} = \frac{B_2}{p} \tag{26}$$

which gives us the reason for finding only B_2 in the last section.

If now we use Hankel's inversion theorem on the exact boundary conditions in equations (9) and substitute for B_2 of equation (26) using the results of equation (24), we ultimately find that the exact governing dual integral equations for the rigid circular body are

$$\left.\begin{aligned} \frac{\Omega_2^2}{G}\int_0^\infty \frac{\alpha_1}{\phi(p,h)}\overline{\sigma(r)}pJ_0(pr)\,\mathrm{d}p &= w_0 \quad (0 < r < R) \\ \int_0^\infty \overline{\sigma(r)}pJ_0(pr)\,\mathrm{d}p &= 0 \qquad (r > R) \end{aligned}\right\}. \tag{27}$$

These equations have been derived for the first time to give an exact formulation of the mixed boundary-value problems. It is pertinent to quote the corresponding equations if we have assumed that the base of the stratum is fixed to the foundation. We find, following the same procedure as above, that the governing dual integral equations are

$$\left.\begin{aligned} \frac{\Omega_2^2}{G}\int_0^\infty \frac{\alpha_1 f_1(h)\overline{\sigma(r)}pJ_0(pr)\,\mathrm{d}p}{(2p^2-\Omega_2^2)^2 f_2(h)-4p^2\alpha_1\alpha_2 f_3(h)} &= w_0 \quad (0 < r < R) \\ \int_0^\infty \overline{\sigma(r)}pJ_0(pr)\,\mathrm{d}p &= 0 \qquad (r > R) \end{aligned}\right\}. \tag{27i}$$

where

$$f_1(h) = p^2\coth(\alpha_1 h)-\alpha_1\alpha_2\coth(\alpha_2 h)$$

$$f_2(h) = p^2-\alpha_1\alpha_2\coth(\alpha_1 h)\coth(\alpha_2 h)$$

$$f_3(h) = p^2\coth(\alpha_1 h)\coth(\alpha_2 h)-\alpha_1\alpha_2-\frac{(\alpha_2^2+p^2)}{\mathrm{sh}(\alpha_1 h)\,\mathrm{sh}(\alpha_2 h)}$$

so that

$$\lim_{h\to\infty} f_{1,2,3}(h) = p^2-\alpha_1\alpha_2.$$

We notice that both equations (27) and (27i) can be shown to reduce exactly to the semi-infinite medium governing equations as the depth h tends to infinity. The case of fixed base is obviously more difficult than the frictionless base and is outside the scope of the present work. We continue in the rest of this paper the frictionless case which is the one discussed by Warburton [1] and Arnold *et al.* [2].

We only need to quote the governing dual integral equations for the rectangular body corresponding to those of equations (27)

$$\left.\begin{aligned} \frac{\Omega_2^2}{\pi G}\int_0^\infty \frac{\alpha_1}{\phi(p,h)}\overline{\sigma(x)}\sin(px)\,\mathrm{d}p &= x\psi \quad (0 < x < b) \\ \int_0^\infty \overline{\sigma(x)}\sin(px)\,\mathrm{d}p &= 0 \qquad (x > b) \end{aligned}\right\} \tag{27a}$$

where ψ is the harmonically varying angle of rock.

5. APPROXIMATE GOVERNING DUAL INTEGRAL EQUATIONS AND THE ESTABLISHMENT OF EQUIVALENT SEMI-INFINITE MEDIUM

No exact solution of the pair of equations (27) is, at present, possible. Although a very tedious point-by-point calculation is possible using a scheme of successive approximations similar to those employed in [11] and involving determination of the poles of the modified Rayleigh function for various values of stratum depth, the procedure will not give an easy interpretation of the general behaviour of the system for all the parameters. The experience of the author in [3] has shown that a useful method of studying the behaviour of the stratum problem is to seek an approximation to the integrand in the first of the dual integral equations. It is very important here to note that, as long as the range of vibrations is limited to low frequency factors, i.e. $\eta_2 < 1$, our approximation will be acceptable if it is close to the exact integrand for large values of the integrating parameter ($\eta > 1$) and exactly equal to it as η tends to infinity. The argument here is that for low frequency factor vibrations the main contributions to the exact integral come from values of $\eta > \eta_2$.

The non-dimensional form of equation (27) is given by

$$\left.\begin{aligned} \frac{\eta_2^2}{G}\int_0^\infty \frac{\alpha_1 F(\eta) J_0(\eta\tilde{r})\,d\eta}{(2\eta^2-\eta_2^2)^2\operatorname{cth}(\alpha_1\tilde{h}) - 4\eta^2\alpha_1\alpha_2\operatorname{cth}(\alpha_2\tilde{h})} &= w_0 \quad (0 < \tilde{r} < 1) \\ \int_0^\infty F(\eta) J_0(\eta\tilde{r})\,d\eta &= 0 \quad (\tilde{r} > 1) \end{aligned}\right\} \tag{28}$$

in which we have introduced the parameters:

$$\eta = pR, \qquad \eta_1 = \Omega_1 R, \qquad \eta_2 = \Omega_2 R, \qquad \tilde{r} = \frac{r}{R}, \qquad \tilde{h} = \frac{h}{R}$$

and

$$F(\eta) = p\overline{\sigma(r)}.$$

We now observe that the double-valued function $\alpha_2 = \sqrt{(\eta^2-\eta_2^2)}$ has been modified in the same manner as in the case of the work in [3] to the form

$$\alpha_2 \coth(\alpha_2\tilde{h}) = \{\sqrt{(\eta^2-\eta_2^2)}\}/\{\tanh[\sqrt{(\eta^2-\eta_2^2)}\tilde{h}]\}$$

which we have found approximates throughout the range $\eta > \eta_2$ to the form

$$\alpha_2\coth(\alpha_2\tilde{h}) \simeq \sqrt{(\eta^2-\eta_{2e}^2)}$$

where

$$\eta_{2e} = (\eta_2^2 - 1/\tilde{h}^2)^{\frac{1}{2}} \tag{29}$$

defines an equivalent frequency factor.

We also observe that the double-valued functions $\alpha_1 = \sqrt{(\eta^2-\eta_1^2)}$ is *not directly* modified. However, its effect in $\coth(\alpha_1\tilde{h})$ is to increase the denominator in the range $\eta > \eta_2$ and, thereby, reduce the integrand. This "misplaced" occurrence of $\coth(\alpha_1\tilde{h})$ and the appearance of the other two η_2^2 will be accommodated in the approximate integrand by changing every η_2 to η_{2e} and η_1 to η_{1e} where η_{2e} has been defined in equation (29) but η_{1e} is yet undefined.

We now impose a necessary condition on the approximate integrand in order to make it follow closely the exact integrand at least for low frequency factor vibration, i.e. the two integrands must be equal for large values of the integrating parameter, $\eta > \eta_2$. We find below that this condition necessarily defines η_{1e}.

From the foregoing, we postulate an equivalent semi-infinite elastic medium vibrating at shear and dilatational frequency factors η_{2e} and η_{1e}, respectively. The governing dual integral equations comparable with those of equations (28) are, therefore

$$\left.\begin{aligned} \frac{\eta_{2e}^2}{G}\int_0^\infty \frac{\sqrt{(\eta^2-\eta_{1e}^2)}F(\eta)J_0(\eta\tilde{r})\,d\eta}{(2\eta^2-\eta_{2e}^2)^2-4\eta^2\sqrt{(\eta^2-\eta_{1e}^2)}\sqrt{(\eta^2-\eta_{2e}^2)}} &= w_0 \quad (0<\tilde{r}<1) \\ \int_0^\infty F(\eta)J_0(\eta\tilde{r})\,d\eta &= 0 \quad (\tilde{r}>1) \end{aligned}\right\}. \tag{30}$$

As mentioned above, in order to ascertain that the first of equations (30) is a fair approximation to its counterpart in equations (28) at least for low frequency factor vibrations, both integrands must be equal as the integrating parameter η tends to infinity. It is easy to check from asymptotic expansions that for large values of η

$$\eta_2^2\alpha_1\{(2\eta^2-\eta_2^2)^2\coth(\alpha_1\tilde{h})-4\eta^2\alpha_1\alpha_2\coth(\alpha_2\tilde{h})\}^{-1} \simeq -\frac{1}{2\eta\left(1-\dfrac{\eta_1^2}{\eta_2^2}\right)} \tag{28i}$$

and

$$\eta_{2e}^2\sqrt{(\eta^2-\eta_{1e}^2)}\{(2\eta^2-\eta_{2e}^2)^2-4\eta^2\sqrt{(\eta^2-\eta_{1e}^2)}\sqrt{(\eta^2-\eta_{2e}^2)}\}^{-1} \simeq -\frac{1}{2\eta\left(1-\dfrac{\eta_{1e}^2}{\eta_{2e}^2}\right)}. \tag{30i}$$

Hence, we immediately conclude that equations (30) are approximately equal to equations (28) if, and only if

$$\frac{\eta_{1e}}{\eta_{2e}} = \frac{\eta_1}{\eta_2} \tag{31}$$

which now defines η_{1e}.

It is important here to note the remarkable consequence of equation (31) because

$$\frac{\eta_1^2}{\eta_2^2} = \frac{c_2^2}{c_1^2}$$

is a single-valued function of Poisson's ratio ν in accordance with the relation

$$\frac{c_2^2}{c_1^2} = \frac{1-2\nu}{2(1-\nu)}. \tag{32}$$

Thus the direct consequence of equation (31) is that the frequency factors or the wave velocities have been reduced in the same proportion thus implying no change of Poisson's ratio. We, therefore, conclude that a semi-infinite elastic medium vibrating at a lower frequency factor η_{2e} and of the same elastic constants is approximately equivalent to the above stratum vibrating at the frequency factor η_2. The approximation is valid for all values of $\eta_2 > 1/\tilde{h}$ and improves with increasing $\tilde{h}$.

We finally conclude our justification of the fairness of the approximate dual integral equations by comparing in Table 1 the above expressions on the left-hand sides of equations (28i) and (30i) even for as large a value as $\eta_2 = 1$ for strata of depths $\tilde{h} = 2, 3, 5$ and the whole range of Poisson's ratio by choosing $\nu = 0, \frac{1}{4}$ and $\frac{1}{2}$. The closeness of the two expressions even at values of the integrating parameter as low as $\eta = 3$ and 2 shows how accurate the approximate integrand is for larger values of η. We should note, however, that the only limitation to the approximation is that $\eta_{2e} > 0$ or $\eta_2 > 1/\tilde{h}$.

Therefore, we have established an approximate equivalent semi-infinite medium of the same Poisson's ratio and shear modulus as the stratum but only vibrating at a reduced frequency factor η_{2e}.

TABLE 1

η	$\tilde{h}$	ν	Exact expression	Approximate expression
		0	0·36	0·36
	5	$\frac{1}{4}$	0·27	0·27
		$\frac{1}{2}$	0·18	0·18
		0	0·36	0·36
3	3	$\frac{1}{4}$	0·27	0·27
		$\frac{1}{2}$	0·18	0·17
		0	0·36	0·36
	2	$\frac{1}{4}$	0·27	0·26
		$\frac{1}{2}$	0·18	0·17
		0	0·62	0·62
	5	$\frac{1}{4}$	0·44	0·44
		$\frac{1}{2}$	0·29	0·28
		0	0·62	0·61
2	3	$\frac{1}{4}$	0·44	0·43
		$\frac{1}{2}$	0·29	0·28
		0	0·62	0·59
	2	$\frac{1}{4}$	0·44	0·42
		$\frac{1}{2}$	0·29	0·28

6. AMPLITUDE RESPONSE

We now consider the motion of the rigid body under the action of a sinusoidal force $F = \hat{F}\,e^{i\Omega t}$ its inertia and the stress distribution integrated over the circular base. It is easy to show that the amplitude response, $\hat{w}_0$ of the body is governed by an equation of the form

$$GR\hat{w}_0 f(\eta_{2e}) - m\Omega^2\hat{w}_0 = \hat{F} \tag{33}$$

where $f(\eta_{2e})$ represents the integration of the dynamic stress distribution which consists of a real part representing the "spring" stiffness and a quadrature component representing the dispersion of waves and producing damping in the medium.

If we introduce the relations

$$\eta_2^2 = \frac{R^2\Omega^2\rho}{G}, \qquad m = \tilde{m}\rho R^3$$

in which ρ is the stratum density, we find that

$$\hat{\hat{w}}_0 = \frac{GR\hat{w}_0}{\hat{F}} = \frac{1}{f(\eta_{2e}) - \left(\tilde{m}\dfrac{\eta_2^2}{\eta_{2e}^2}\right)\eta_{2e}^2}$$

$$= \frac{1}{f(\eta_{2e}) - \tilde{m}_e \eta_{2e}^2} \tag{34}$$

where

$$\tilde{m}_e = \tilde{m}\left(\frac{\eta_2^2}{\eta_{2e}^2}\right) \tag{35}$$

is an equivalent mass ratio greater than the actual mass ratio $\tilde{m}$ but vibrating at a lower frequency factor η_{2e} on the equivalent semi-finite medium. Thus, the amplitude $\hat{\hat{w}}_0$ of mass $\tilde{m}$ vibrating at η_2 on the stratum is the same as that of the equivalent mass $\tilde{m}_e$ vibrating at η_{2e} on the semi-infinite medium. Therefore, for any stratum system, we only need to find the equivalent quantities m_e, η_{2e} and read off the amplitude $\hat{\hat{w}}_0$ from the equivalent semi-infinite medium.

It is now left to give the response curves for determining the modulus $|\hat{\hat{w}}_{0e}|$ of the amplitude on the equivalent semi-infinite medium. The theory for this has been given in a previous work [11] where only a few resonance curves for only zero Poisson's ratio were worked out. The need arises here to cover the whole range of Poisson's ratio and the curves for Poisson's ratios of 0, $\frac{1}{4}$, $\frac{1}{3}$ and $\frac{1}{2}$ are given in Fig. 1 for a wide range of mass ratios which should cover most practical cases. Amplitudes in intermediate range of Poisson's ratio can be found by cross-plotting or interpolating. These curves have been generated from the equations

$$|\hat{\hat{w}}_{0e}| = \frac{1}{\sqrt{(P_e^2 + Q_e^2)}} \tag{36}$$

where Table 2 gives algebraic expressions for the real part P_e and the quadrature component Q_e for each of the four values of Poisson's ratio. These results are being given for the first time but they have been checked in part by agreement with the experimental results of Arnold *et al.* [2].

The response for the rigid rectangular body follows similar analysis except for replacing the mass ratio $\tilde{m}$ by the polar inertia ratio $\tilde{J}$ about the axis of rock to obtain expression for the non-dimensional angle of rock $\hat{\hat{\psi}}$. The equivalent inertia $\tilde{J}_e$ is greater than $\tilde{J}$ similarly as in equation (35). The complete resonance curves for the rocking rectangular body corresponding to those in Fig. 1 on the equivalent semi-infinite medium have been given in a previous work [10] as earlier mentioned.

7. DISCUSSION OF RESULTS AND COMPARISON WITH EXPERIMENTS

The results in the last section have been used to consider answers to the two fundamental questions posed in the introduction. We have generated resonance curves in Figs. 2 and 3 for the two extreme cases of Poisson's ratio $\nu = 0$ and $\frac{1}{2}$. The figures clearly

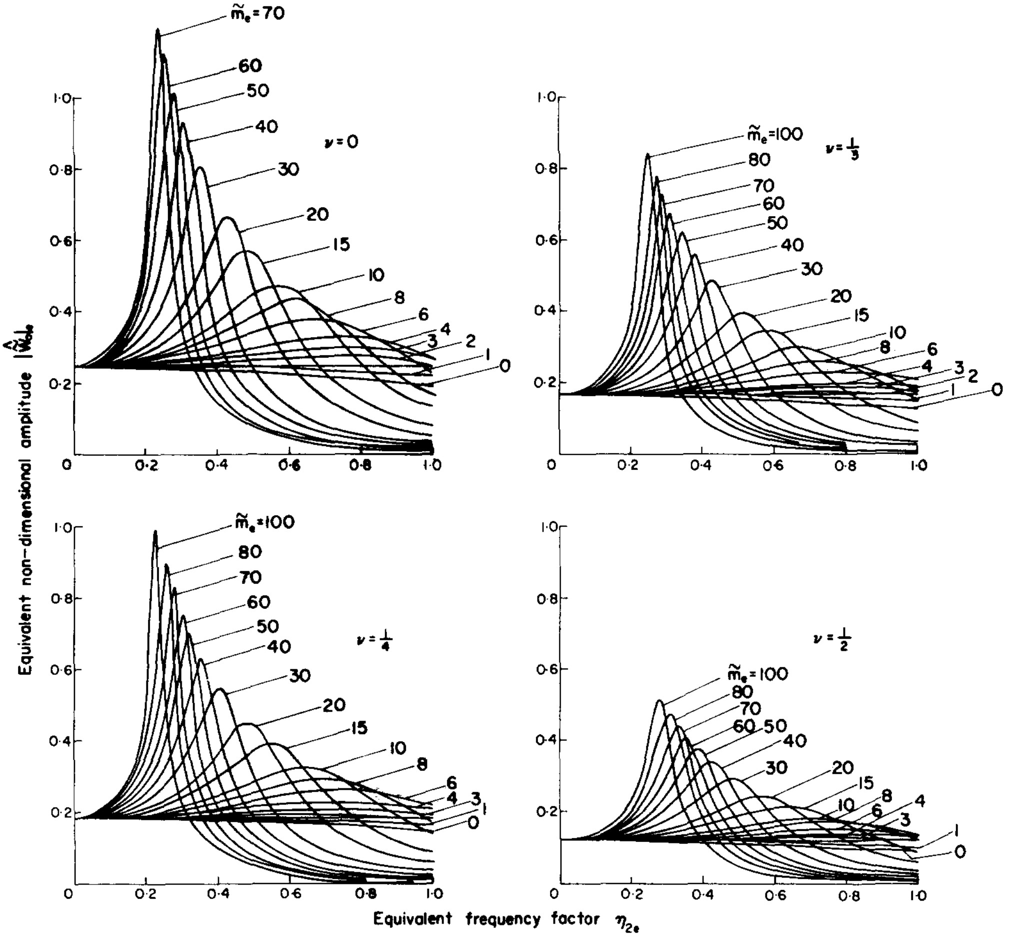

FIG. 1. Resonance curves for vertical vibration of rigid circular body on the equivalent semi-infinite media for Poisson's ratio $\nu = 0, \frac{1}{4}, \frac{1}{3}$ and $\frac{1}{2}$.

TABLE 2

$\nu = 0$	P_e	$4-\eta_{2e}^2(\tilde{m}_e+0{\cdot}649-0{\cdot}122\eta_{2e}^2)$
	Q_e	$3{\cdot}46\eta_{2e}(1+0{\cdot}0506\eta_{2e}^2)$
$\nu = \frac{1}{4}$	P_e	$5\frac{1}{3}-\eta_{2e}^2(\tilde{m}_e+0{\cdot}763-0{\cdot}236\eta_{2e}^2)$
	Q_e	$4{\cdot}38\eta_{2e}(1+0{\cdot}050\eta_{2e}^2)$
$\nu = \frac{1}{3}$	P_e	$6-\eta_{2e}^2(\tilde{m}_e+0{\cdot}793-0{\cdot}206\eta_{2e}^2)$
	Q_e	$4{\cdot}74\eta_{2e}(1+0{\cdot}047\eta_{2e}^2)$
$\nu = \frac{1}{2}$	P_e	$8-\eta_{2e}^2(\tilde{m}_e+1{\cdot}667-0{\cdot}288\eta_{2e}^2)$
	Q_e	$6{\cdot}78\eta_{2e}(1+0{\cdot}016\eta_{2e}^2)$

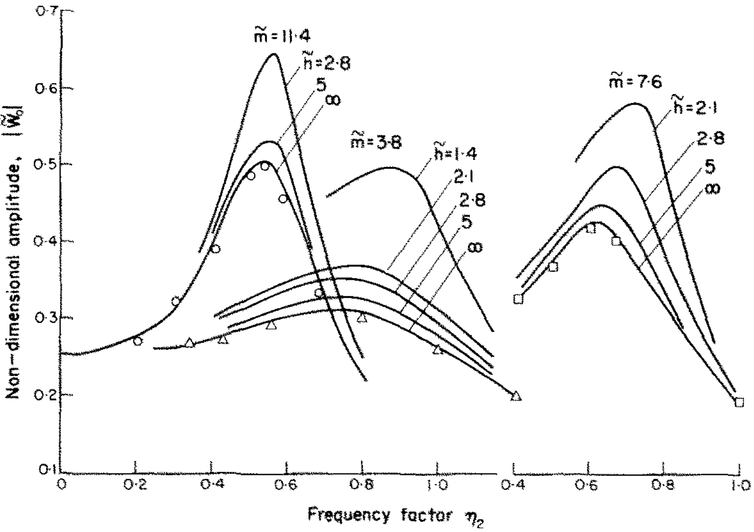

FIG. 2. Resonance curves for vertical vibration of rigid circular body on strata of increasing depths for Poisson's ratio $\nu = 0$ and mass ratios $\tilde{m} = 3{\cdot}8$, $7{\cdot}6$ and $11{\cdot}4$: ———, present theory; ○ △ □, experimental results of Arnold *et al.* [2]; $\tilde{h} = \infty$.

show the effect of mass ratio, Poisson's ratio and stratum depth on the resonance curves. The expected decrease in amplitudes with decreasing mass ratio or increasing Poisson's ratio for a given stratum depth is observed; so also is the asymptotic approach of the strata curves to the semi-infinite media as the depths increase.

We observe that resonant frequency decreases slightly, whereas amplitude decreases appreciably with increasing stratum depth. This agrees with the expected result because increasing depth provides greater damping due to dispersion of waves and consequently peak amplitude occurs before the undamped resonant frequency is attained. However, beyond a certain stratum depth the amount of dispersion becomes negligible and the curves show that for practical purposes a stratum depth of five times the radius of the circular base is sufficient to provide damping that would make the stratum a fair approximation to a semi-infinite medium.

Although the above discussion arises from consideration of the extreme cases of Poisson's ratio similar results are true for the whole range since the approximate integrand is generally applicable provided $\eta_2 > 1/\tilde{h}$ which is the only restraint on the validity of our approximations. It is also useful to note that if the dotted curve in Fig. 5 of Warburton [1] has been superposed on Fig. 6 of his paper it will be seen that all the experimental points for resonant frequency factors of bodies with different mass ratios and on strata of increasing depths as shown in Fig. 6—obtained from Arnold *et al.* [2]—consistently lie to the right of the dotted curve representing the semi-infinite medium limit. Thus these experiments confirm the asymptotic approach of the strata to the semi-infinite medium and, therefore, show that the crossing of strata curves by semi-infinite medium curve in Figs. 5 and 7 of Warburton has no experimental justification.

The results of the present theory have been compared with the experiments of Arnold *et al.* [2] as shown in Fig. 2 and also in Table 3 below. The mass ratios $\tilde{m}$ and depth ratios $\tilde{h}$ used in Fig. 2 are such that it is necessary to separate the middle mass ratio curves, i.e.

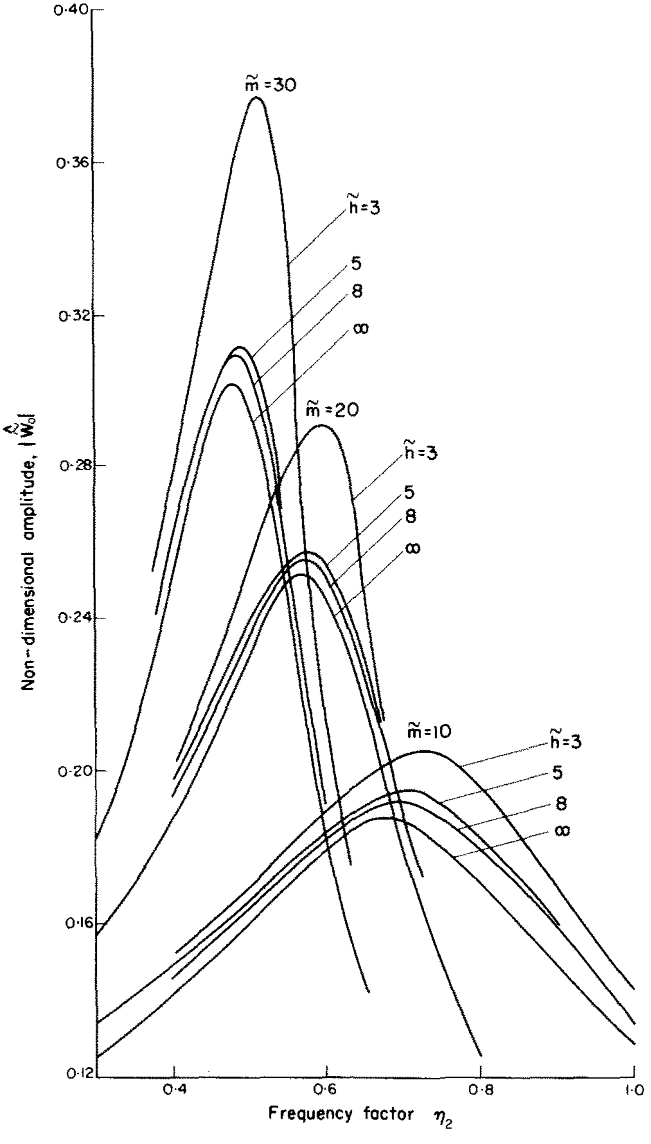

FIG. 3. Resonance curves for vertical vibration of rigid circular body on strata of increasing depths for Poisson's ratio $\nu = \frac{1}{2}$ and mass ratios $\tilde{m} = 10, 20$ and 30.

$\tilde{m} = 7{\cdot}6$ in order to make the curves clearer. The experimental points are those of the semi-infinite medium as shown in Fig. 4(a) of Arnold *et al.* [2]. The asymptotic approach of the strata curves to the semi-infinite medium is thus experimentally confirmed.

We also compare in Table 3 below the resonant frequency factors of these masses on strata of the same depth ratios $\tilde{h}$ as those in the experimental points in Fig. 6 of Warburton [1] with the frequency factors at peak amplitudes in Fig. 2 of the present work. The good agreement justifies the fairness of the approximation to the governing dual integral equations.

TABLE 3

$\bar{m}$	$\bar{h}$	Resonant frequency factor	
		Experiment Arnold *et al.*	Present theory
	1·4	1·0	0·91
3·8	2·1	0·88	0·89
	2·8	0·75	0·75
	∞	0·72	0·72
	2·1	0·73	0·73
7·6	2·8	0·67	0·67
	∞	0·62	0·62
11·4	2·8	0·56	0·56
	∞	0·54	0·54

Finally, it is useful to discuss qualitatively how, in practical terms, our equivalent semi-infinite medium produces the same amplitude response as the stratum. This can be found by first considering frequency factors below resonance where the amplitude increases with increasing frequency factor. Thus an increased inertia will provide the effect of increasing amplitude if our semi-infinite medium is vibrating at the same frequency factor η_2 as the stratum. Therefore, the reduced frequency factor η_{2_e} at which the semi-infinite medium is vibrating produces the compensating effect of reducing the amplitude to that on the stratum. Similarly, this compensating effect is produced beyond resonance as can be seen from Fig. 1 which shows a reversal of the above behaviour of resonance curves: increased inertia now reduces amplitude at a given frequency factor whereas reduced frequency factor η_{2_e} of the semi-infinite medium has the compensating effect of increasing amplitude on this side of the resonance curve. Thus, in practical terms, it is this compensating effect between increased inertia on the one hand and reduced frequency factor on the other that makes our semi-infinite medium a reasonable equivalent of the stratum system.

The above discussion applies also to the rocking of the rigid rectangular body as shown in Figs. 4 and 5. We find also from all the above comments that for large inertia ratio bodies, greater than about 10, the response of the body can be estimated from the semi-infinite medium solution irrespective of the stratum depth.

8. CONCLUSIONS

The mixed boundary-value problems of the vertical vibration of a rigid circular body and of the rocking of a long rigid rectangular body on an infinitely wide elastic stratum have been precisely formulated in terms of dual integral equations. It is found that, for the case of a frictionless contact base with foundation, the rigid body on a stratum can be reduced approximately to a body with increased inertia vibrating at a lower frequency factor on a semi-infinite medium of the same elastic constants. The governing dual integral equations for the case of a fixed base are given but not solved.

The work finally answers the two questions asked in the introduction by showing that resonant frequency factor reduces with increasing stratum depth and that a stratum

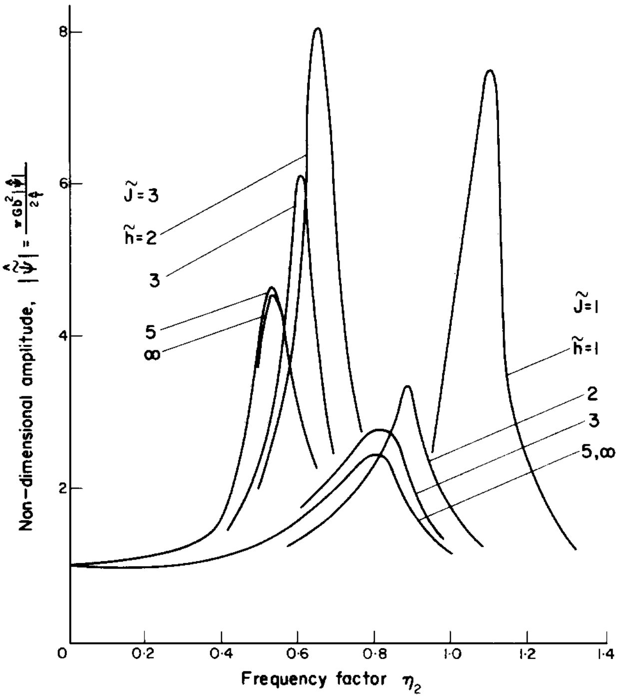

FIG. 4. Effect of inertia ratio $\tilde{J}$ and depth $\tilde{h}$ on resonance curves for rocking of rigid rectangular body on a stratum of Poisson's ratio $\nu = 0$.

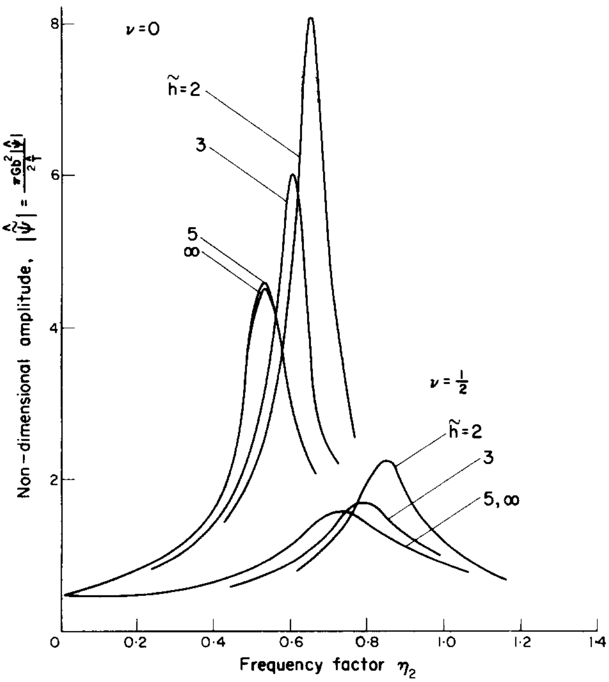

FIG. 5. Effect of Poisson's ratio and stratum depth $\tilde{h}$ on resonance curves for rocking of rigid rectangular body of inertia ratio $\tilde{J} = 3$.

with a depth of about five times the circular radius is a very fair approximation to a semi-infinite medium. The results agree with the experimental results of Arnold *et al.* [2] and provide a faster prediction of a stratum behaviour by comparison with the semi-infinite medium in which comprehensive resonance curves have been provided over the whole range of Poisson's ratio. The work finally discusses the practical interpretation and justification for the equivalence of the semi-infinite medium system and the given stratum system.

REFERENCES

[1] G. B. Warburton, Forced vibration of a body on an elastic stratum. *J. appl. Mech.* **24**, 55 (1957).

[2] R. N. Arnold, G. N. Bycroft and G. B. Warburton, Forced vibrations of a body on an infinite elastic solid. *J. appl. Mech.* **22**, 391 (1955).

[3] A. O. Awojobi, Torsional vibration of a rigid circular body on an infinite elastic stratum. *Int. J. Solids Struct.* **5**, 369 (1969).

[4] G. M. L. Gladwell, The forced torsional vibration of an elastic stratum. *Int. J. Engng Sci.* **7**, 1011 (1969).

[5] H. S. Paul, Longitudinal wave propagation in an infinite isotropic elastic plate with a penny-shaped crack. *Proc. Camb. phil. Soc. math. phys. Sci.* **66**, 439 (1969).

[6] H. S. Paul and N. Muthiyalu, Sinusoidal rocking of a rigid rectangular body on an infinite isotropic elastic plate. *J. appl. Mech.* **36**, 648 (1969).

[7] H. S. Paul and N. Muthiyalu, Vibration in an infinite isotropic elastic plate due to a vertical harmonic load. *J. acoust. Soc. Am.* **46**, 191 (1969).

[8] P. Grootenhuis and A. O. Awojobi, The *in-situ* determination of the dynamic properties of soil, *Proc. Int. Conf. on Earthquake Engineering*, Imperial College, London (1965); published in *Vibrations in Civil Engineering*, p. 181. Butterworths (1965).

[9] A. O. Awojobi, Determination of the dynamic shear modulus and the depth of the dominant layer of a vibrating elastic medium. *Int. J. Solids Struct.* **6**, 315 (1970).

[10] A. O. Awojobi, Harmonic rocking of a rigid rectangular body on semi-infinite elastic medium. *J. appl. Mech.* **33**, 547 (1966).

[11] A. O. Awojobi and P. Grootenhuis, Vibration of rigid bodies on semi-infinite elastic media. *Proc. R. Soc.* **A287**, 27 (1965).

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Абстракт—Предлагаются точные формулы, в виде парных интегральных уравнений, касающиеся смешаных задач на краевые условия, для вертикального колебания жесткого, круглого тела и качания длинного, жесткого, прямоугольного тела, на бесконечно широком упругом слое. Для случая основания фундамента, без учёта трения, получаются приближение решения, путем учреждения, новым способом, эквивалентной системы на полубесконечной упругой среде. Оказывается, что поведение тела, колеблющегося при факторе частоты η_2 на слое конечной толщины прибизительно эквивалентное поведению такого-же тела, которого инерция увелцчена фактором η_2^2/η_{2e}^2, но оно колеблется при более низким факторе частоты $\eta_{2e} = (\eta_2^2 - \bar{n}/\bar{h}^2)^{1/2}$ на полубесконечной среде, обладающей этими же самыми упругими постоянными, как слой безразмерной толщины $\bar{n}$. Все результаты приближения соответствуют результатам для полубсконечной среды, сели толщина слоя стремится к бесконечностн. Этот факт, затем, исправляет погрешность Уарбартона [1], в которой поведение тела на полубесконечной среде находится между поведениями на слое конечной толщины, в противоположность ожидаемому ассимптотическому приближению, проберениому экспериментами Арнолда, Ьейкрофа и Уарбартона [2].

В заключение, установливаются два важные результаты для такой системы: толщина слоя приблизительно пять раз больше радууса осноиания/или половины толщины для прямоугольного тела/является очень хорошим приблжением для полубесконечной среды; частота резонанса тела на слое уменьшается при росте толщины этого же слоя. Далее, фактор частоты резонанса, η^2, для тел с большими отношениями инерции/больше чем около десяти/, можно определить из решения полубесконечной среды, независимо от толщины слоя. Предлагаемая теория указывает надлежашую постоянную сходимость с экспериментальными результатами Арнолда и др. [2].